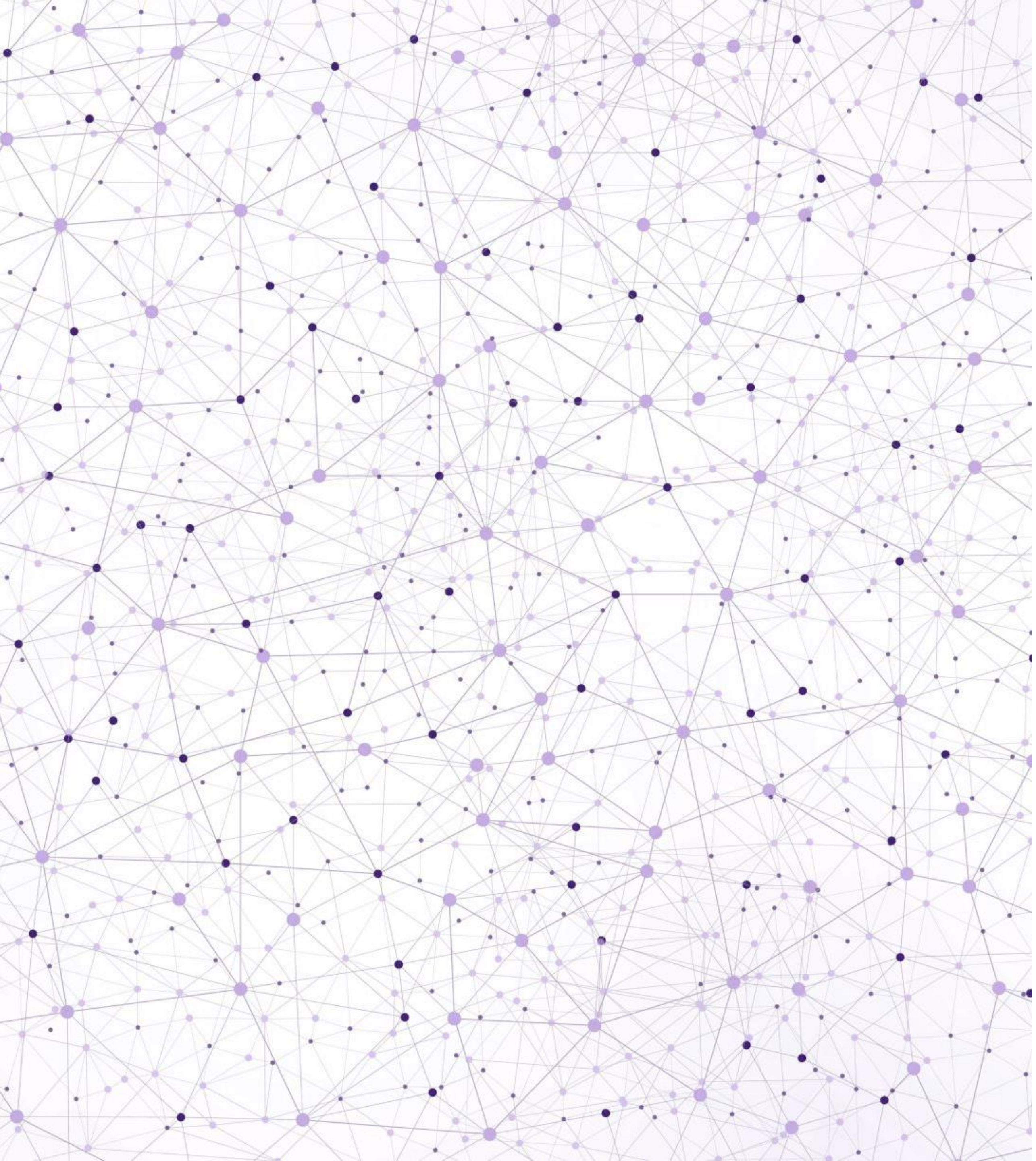

EULER

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INTRODUCTION

- Leonhard Euler (15th of April 1707 -18th of September 1783) was one of the most influential and important people in math, physics, astronomy, and engineering amongst other things. He discovered infinitesimal calculus and graph theory as well as pioneered and led several branches such as analytic number theory and topology. Many of the terms that we use in mathematics are discovered by him, such as employing the function of y to be written as f(x); using sigma $\sum$ to symbolize summation; using "***i***" to express the imaginary number $\sqrt{-1}$;and developing "**e**" also known as Euler's number which is roughly equal to 2.71828. His contributions are considered some of the greatest. Three of his equations were given a place in the top 5 most beautiful equations, granting his equation "Euler's identity" the top 1 spot (figure 2).

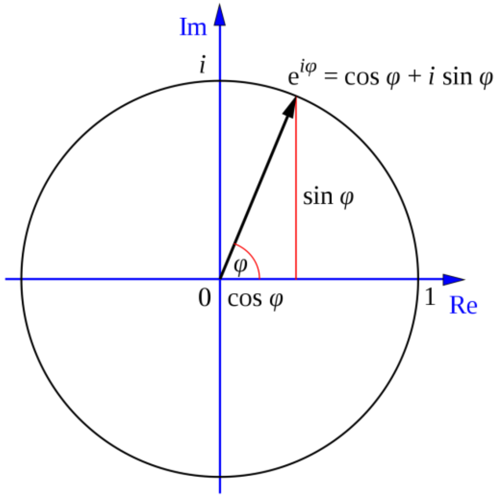


Figure 3: A geometric interpretation of Euler's formula

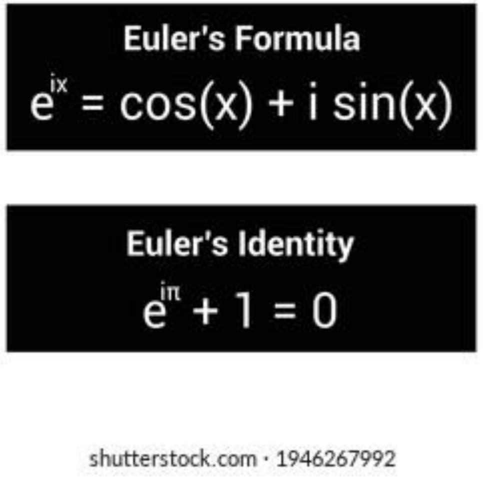


Figure 2: Euler's Formula and Euler's Identity

Figure 1:Portrait by Jakob Emanuel Handmann (1753)

Euler’s formula

Euler's Formula

$e^{ix} = \cos(x) + i\sin(x)$

In order to prove Euler’s formula, we are going to use the Maclaurin series, which is the Taylor series (shown below) with a = 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If the Maclaurin series is expanded, it can be rewritten as :

$$\boldsymbol{f(x) = \ f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}}$$

If we consider $f(x) = \ e^x \ and \ f'(x) = \ e^x \ and \ f''(x) = e^x \ and \ f^{(n)}(x) = e^x \ and \ f(0) = e^0 = 1 \ \ then:$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{(first equation)}$$

Euler's Formula

$e^{ix} = \cos(x) + i\sin(x)$

We can use the Maclaurin series to describe both sin(x) and cos(x):

$$\boldsymbol{f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}}$$

If we consider $\boldsymbol{f(x) = \sin(x)}$ and $f(0) = \sin(0) = 0$

$$f'(0) = (\sin(0))' = \cos(0) = 1$$

$$f''(0) = (\sin(0))'' = -\sin(0) = 0$$

$$f^{(3)}(0) = (\sin(0))^{(3)} = -\cos(0) = -1$$

We can conclude that:

$$\boldsymbol{f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}}$$

Is equivalent to the following:

$$\sin(x) = 0 + (1)x + \frac{(0)x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{(0)x^4}{4!} + \frac{(1)x^5}{5!} + \cdots$$

$$\boldsymbol{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots}$$

If we consider $\boldsymbol{f(x) = \cos(x)}$ and $f(0) = \cos(0) = 1$

$$f'(0) = (cos(0))' = -\sin(0) = 0$$

$$f''(0) = (\cos(0))'' = -\cos(0) = -1$$

$$f^{(3)}(0) = (\cos(0))^{(3)} = \sin(0) = 0$$

We can conclude that:

$$\boldsymbol{f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}}$$

Is equivalent to the following:

$$\cos(x) = 1 + (0)x + \frac{(-1)x^2}{2!} + \frac{(1)x^4}{4!} + \frac{(-1)x^6}{6!} + \frac{(1)x^8}{8!} + \cdots$$

$$\boldsymbol{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots}$$

We can rewrite the (first equation) of e to equal:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Euler's Formula

$e^{ix} = \cos(x) + i \sin(x)$

$$e^x = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)$$

Afterwards we substitute $x = ix$ which will give us

$$e^{ix} = \left(1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \frac{(ix)^6}{6!} + \cdots\right) + \left(ix + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \frac{(ix)^7}{7!} + \cdots\right)$$

Using the special property of $i^2 = -1$ *and* $i^4 = i^2 * i^2 = 1$ and so on, we can convert the previous equation to be equal to

$$e^{ix} = \left(1 + \frac{i^2x^2}{2!} + \frac{i^4x^4}{4!} + \frac{i^6x^6}{6!} + \cdots\right) + \left(ix + \frac{i * i^2x^3}{3!} + \frac{i * i^4x^5}{5!} + \frac{i * i^6x^7}{7!} + \cdots\right)$$

$$e^{ix} = \left(1 + \frac{(-1)x^2}{2!} + \frac{(1)x^4}{4!} + \frac{(-1)x^6}{6!} + \cdots\right) + \left(ix + \frac{(-1)ix^3}{3!} + \frac{(1)ix^5}{5!} + \frac{(-1)ix^7}{7!} + \cdots\right)$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots\right)$$

(second equation)

Euler's Formula

$e^{ix} = \cos(x) + i\sin(x)$

Now we factor out the i in the 2nd term of the (second equation), for us to receive:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

If we look closely, we notice that the 1st and 2nd terms are respectively equal to the cos(x) and sin(x) Maclaurin series expansions given in (2) and (3) equations.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

Therefore, we can substitute the 1st term with cos(x) and the 2nd term with sin(x) in the end to receive

$$e^{ix} = \cos(x) + isin(x)$$

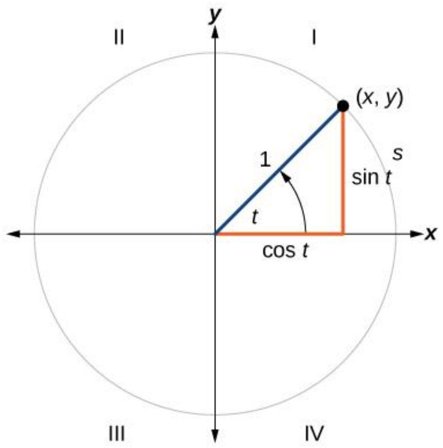
And to conclude this we can turn the Euler Formula into the Euler Identity using basic trigonometry and arithmetic as follows

Euler's Identity

$e^{i\pi} + 1 = 0$

First, we substitute x with π

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$



And we know that the values of cosine and sine are

$$\cos(\pi) = -1 \quad \text{and} \quad \sin(\pi) = 0$$

Now we plug these values into our equation

$$e^{i\pi} = -1 + i * 0$$

If we add one to both sides of the equation, we will finally be given what is considered to be one of the most beautiful equations in math

$$e^{i\pi} + 1 = 0$$

An equation that connects seemingly unrelated terms such as i an irrational number, π a number that intrigued scientists throughout history and is related to circles, and Euler's number all in a single equation with 1, resulting in 0.

CONCLUSION

- Our high school professor had already introduced us to Euler’s Formula, yet I have always wondered where it came from or how to prove it. After doing this research I realized how much of a great person Euler is, having discovered and invented various terms that we consider to be obvious, or a given. This project made me believe that human potential can be limitless, and how seemingly unrelated things can be found to be interconnected in the most mind-boggling ways. He inspires me to work hard and strive to be someone great.

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