Interpolatory quadratures

An Introduction

Victor Martínez Gil

McLennan Community College

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Quadrature formulas

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that approximates the value of an integral

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Normally it is assumed that $y_k = f(x_k)$, where

$$a \leq x_0 < x_1 < \cdots < x_n \leq b;$$

points x_k are called nodes and coefficients λ_k are the weights of $\mathcal{I}_n(f)$.

Intepolatory Quadratures: general idea

One easy way to obtain such formulas is to replace the integrand f by a simpler function, say a polynomial q:

$$I(f) = \int_a^b f(x)\,dx \approx \int_a^b q(x)\,dx = \mathcal{I}(f).$$

If this polynomial interpolates f in the given nodes, that is, if

$$q(x_0) = f(x_0),\ q(x_1) = f(x_1),\ \ldots,\ q(x_n) = f(x_n),$$

we say that the resulting quadrature formulas are interpolatory.

Simple trapezoid rule: interpolating with a straight line

In the simplest case we interpolate f at **two nodes**, $x_0 = a$ and $x_1 = b$. The interpolating polynomial is

$$q(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}.$$

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Replacing f by q in the integral we get

$$\begin{aligned}
\mathcal{I}(f) &= \int_a^b q(x)dx = \int_a^b \left(f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a} \right) dx \\
&= \frac{f(a)}{a-b}\int_a^b (x-b)dx + \frac{f(b)}{b-a}\int_a^b (x-a)dx \\
&= \frac{f(a)}{2(a-b)}(x-b)^2\Big|_a^b + \frac{f(b)}{2(b-a)}(x-a)^2\Big|_a^b \\
&= \frac{f(a)}{2(a-b)}(a-b)^2 + \frac{f(b)}{2(b-a)}(b-a)^2.
\end{aligned}$$

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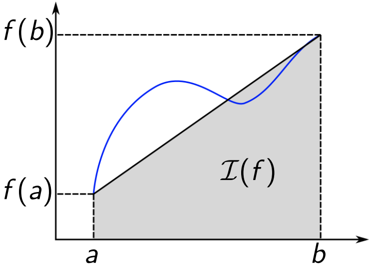
After simplifications we get the following quadrature formula, known as the trapezoid rule:

$$\boxed{\int_a^b f(x)\,dx \approx \mathcal{I}(f) = (b-a)\frac{f(a)+f(b)}{2}}$$

Simple trapezoid rule: interpolating with a straight line

Graphically:



$$\mathcal{I}(f) = (b-a)\frac{f(a)+f(b)}{2}$$

Simple Simpson's rule: interpolating with a parabola

Let us try to do better and interpolate our function f at **3 nodes**:

$$x_0 = a, \quad x_1 = (a+b)/2, \quad x_2 = b.$$

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The interpolating polynomial now is

$$q(x) = f(a)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f\left(\frac{a+b}{2}\right)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(b)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

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As before, replacing f by q in the integral and simplifying we get the formula

$$\boxed{\mathcal{I}(f) = (b-a)\left[\frac{1}{6}f(a) + \frac{4}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right]}$$

known as the Simpson's rule.

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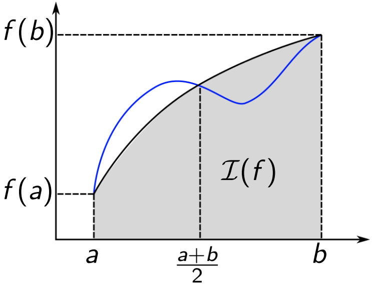


$$\mathcal{I}_n(f) = (b-a)\left[\frac{1}{6}f(a) + \frac{4}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right]$$

Intepolatory Quadratures: general formulas

We can generalize these ideas by interpolating in more nodes. If we define the basic Lagrange polynomials for the nodes $x_0 < x_1 < x_2 < \cdots < x_n$ as

$$\ell_k(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{k-1})(x - x_{k+1}\dots(x - x_n)}{(x_k - x_0)(x_k - x_1)\dots(x_k - x_{k-1})(x_k - x_{k+1}\dots(x_k - x_n)}$$

characterized by the property that

$$\ell_k(x_j) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$q(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + \cdots + f(x_n)\ell_n(x)$$

interpolates f at these nodes.

Intepolatory Quadratures: general formulas

Plugging q instead of f in the integral we get

$$\int_a^b q(x)\,dx = \int_a^b \left(\sum_{k=1}^{n} f(x_k)\ell_k(x)\right)\,dx = \sum_{k=0}^{n} \lambda_k f(x_k) = \mathcal{I}(f),$$

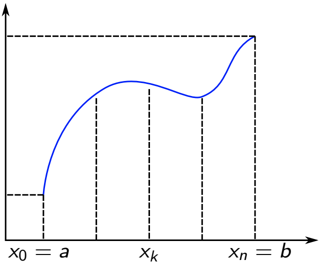
where the weights

$$\lambda_k = \int_a^b \ell_k(x)\,dx, \quad k = 0, 1, \ldots, n.$$

Compound formulas: general idea

Another idea is to partition the interval $[a, b]$ in **equally spaced nodes**,

$$x_k = a + k\Delta x, \quad k = 0, 1, \ldots, n, \quad \Delta x = \frac{b-a}{n}$$



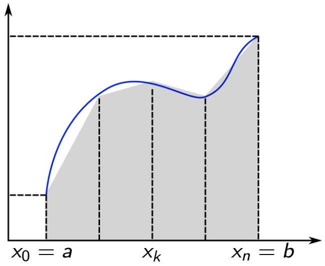
and apply the previous formulas en each subinterval $[x_{k-1}, x_k]$:

$$\mathcal{I}(f) = \mathcal{I}_1(f) + \ldots \mathcal{I}_n(f).$$

Compound trapezoidal rule

If in each subiterval we use the trapezoidal rule, we get

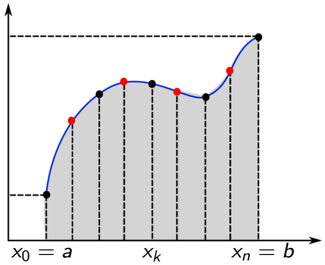
$$\mathcal{I}(f) = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$



Compound Simpson's rule

If in each subiterval we use the Simpson's rule, we get

$$\mathcal{I}(f) = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$



Computer Science Applications

Computers cannot compute integrals directly!

- Numerical integration algorithms replace integrals by finite sums.
- These algorithms can be created by replacing the function f by a "similar" polynomial q.
- In case q is the interpolating polynomial of f, we call such quadrature formulas interpolatory.
- To minimize the error, we subdivide the domain of integration of f into smaller intervals and apply these formulas in each one of them.
- Using the computational power we can apply really fine subdivisions, reaching high precision.

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Thank you for your attention!